

Minimal Generalized Interpolation Projections

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Received October 22, 1975

1. INTRODUCTION AND PRELIMINARIES

The problem of finding projections of minimal norm from $C[0, 1]$ onto the n th degree polynomial subspace $\mathcal{P}_n[0, 1]$ has been investigated by numerous authors, most notably Cheney, Morris, and Price (see, for example [2, 3, 4, 6, 7, 8]). The complete solution to this problem, even in the quadratic case $n = 2$, is unknown. It can be shown (see [1]), however, that among the minimal projections there is a *symmetric* projection P_s , satisfying $P_s[f(\cdot)](x) = P_s[f(1 - \cdot)](1 - x)$, for all $x \in [0, 1]$, $f \in C[0, 1]$.

In this paper we investigate a subclass of these projections, called *generalized interpolating projections* (introduced in [4; Lemma 9]), and determine explicitly the minimal symmetric generalized interpolating projection in the quadratic case $n = 2$. Two representations for the minimum norm are provided below, and computational procedures based on each are illustrated for the quadratic case. Most of the theory in Sections 1 and 2 generalizes to arbitrary Haar subspaces of $C[0, 1]$.

Bounded projections from $C[0, 1]$ onto $\mathcal{P}_n[0, 1]$ can be represented in the form

$$P = \sum_{i=0}^n \mathcal{L}_i \otimes v_i,$$

where $\mathcal{L}_0, \dots, \mathcal{L}_n$ are independent bounded linear functionals on $C[0, 1]$, and $v_0, \dots, v_n \in \mathcal{P}_n[0, 1]$ are determined from $\mathcal{L}_i v_j = \delta_{ij}$ (Kronecker delta), i.e., the \mathcal{L}_i and the v_j form a biorthogonal system.

DEFINITION. A *generalized interpolating projection* from $C[0, 1]$ onto $\mathcal{P}_n[0, 1]$ is a projection which has at least one representation $P = \sum_{i=0}^n \mathcal{L}_i \otimes v_i$ in which the linear functionals $\mathcal{L}_0, \dots, \mathcal{L}_n$ have disjoint supports.

Notation. In the sequel, *generalized interpolating projection* will be abbreviated to *gi projection* or *gip*.

The terminology gi projection is based on the interpolation property:

Given arbitrary constants $|c_i| < \|\mathcal{L}_i\| (i = 0, \dots, n)$, there exists an $f \in C[0, 1]$, $\|f\|_\infty = 1$, which interpolates c_i at \mathcal{L}_i ($\mathcal{L}_i f = c_i$) for $i = 0, \dots, n$.

Then one also has that $Pf = \sum_{i=0}^n (\mathcal{L}_i f) v_i = \sum_{i=0}^n c_i v_i$ is an element of \mathcal{P}_n , in the range of the unit ball of $C[0, 1]$, which interpolates the values c_i at the \mathcal{L}_i . An immediate example of gi projections is given by the *interpolating projections*, i.e., those projections generated by taking $\mathcal{L}_i = e_{x_i}$, point evaluation at x_i , for $i = 0, \dots, n$.

Note. The projection $P = \sum_{i=0}^n \mathcal{L}_i \otimes v_i$ is invariant under an invertible linear transformation of the $(n+1)$ -tuple $\mathcal{L} = (\mathcal{L}_0, \dots, \mathcal{L}_n)$, since, for T a nonsingular $(n+1) \times (n+1)$ matrix, one has

$$P = \mathcal{L} \otimes v = (\mathcal{L}T) \otimes (vT^{-1}), \quad v = (v_0, \dots, v_n),$$

where T' denotes transpose. Thus $P = \mathcal{L} \otimes v$ is a gi projection if and only if there is an invertible transformation T such that the elements of $\mathcal{L}'T$ are "disjoint" linear functionals.

From Cheney and Price [4; Lemma 9], one has the following fact.

PROPOSITION 1. *If $P = \sum_{i=0}^n \mathcal{L}_i \otimes v_i$ is a gi projection, normalized (without loss) so that all $\|\mathcal{L}_i\| = 1$, then*

$$\|P\| = \left\| \sum_{i=0}^n |v_i| \right\|.$$

A second representation for the norm of a gi projection is given in the following statement, which is a simple consequence of Proposition 1.

PROPOSITION 2. *If $P = \sum_{i=0}^n \mathcal{L}_i \otimes v_i$ is a gi projection, normalized (without loss) so that all $\|\mathcal{L}_i\| = 1$, then*

$$\|P\| = \sup \|p_n\|_\infty,$$

where the supremum is taken over all $p_n \in \mathcal{P}_n$ such that $\|\mathcal{L}_i p_n\| = 1$ ($i = 0, \dots, n$).

2. CHARACTERIZATIONS FOR THE INFIMUM OF THE NORMS OF GENERALIZED INTERPOLATION PROJECTIONS

In the previous section two characterizations were given for the norm of a gi projection. These will now be used to develop computationally useful characterizations for the infimum of the norms of gi projections (Theorems 2 and 3). A major additional tool in this development is the following known

"quadrature-formula" type result (see, e.g., [5]). A short proof is given for the benefit of the reader.

THEOREM 1. Let $\hat{\mathcal{L}} \in \mathcal{P}_n^*[0, 1]$ with $\|\hat{\mathcal{L}}\| = 1$, and suppose \mathcal{L} is a norm 1 extension of $\hat{\mathcal{L}}$ to $C^*[0, 1]$.

(a) If \mathcal{L} is signed, then \mathcal{L} is supported on no more than $n + 1$ points (exactly $n + 1$ points implies the points are the Tchebycheff points on $[0, 1]$).

(b) If \mathcal{L} is positive, then \mathcal{L} can be replaced by another norm 1 extension of $\hat{\mathcal{L}}$, \mathcal{L}^* , which is positive and supported on no more than $[(n + 2)/2]$ points.

Proof. (a) Suppose \mathcal{L} is signed and has no fewer than $n + 1$ points in its support. Then $\|\mathcal{L}\| = \|\hat{\mathcal{L}}\| = 1$ implies that \mathcal{L} achieves its norm at some $p \in \mathcal{P}_n$, where $\|p\|_\infty = 1$. But then \mathcal{L} must have all its mass concentrated at the points x_i where $|p(x_i)| = 1$. Since there are at least $n + 1$ such points, p must be the n th degree Tchebycheff polynomial, and \mathcal{L} has exactly the $n + 1$ "Tchebycheff points" as its support.

(b) Suppose \mathcal{L} is positive and supported on more than n points. Then \mathcal{L} gives rise to a positive measure μ on $C[0, 1]$, and induces an inner product $(f, g) = \int_0^1 fg \, d\mu$ on $C[0, 1]$. Let $r = [(n + 2)/2]$. Let x_i ($i = 1, \dots, r$) be the roots of q_r , where q_0, \dots, q_r are the orthogonal polynomials obtained from $1, x, \dots, x^r$ by the Gram-Schmidt orthogonalization process (with respect to the inner product (\cdot, \cdot)). Then the theory of orthogonal polynomials provides that there exist positive numbers a_i ($i = 1, \dots, r$) such that $\mathcal{L}p = \sum_{i=1}^r a_i p(x_i)$ for all $p \in \mathcal{P}_n$. Taking $\mathcal{L}^* = \sum_{i=1}^r a_i \delta_{x_i}$, one has (i) $\mathcal{L} = \mathcal{L}^*$ on \mathcal{P}_n ; and (ii) $\|\mathcal{L}^*\| = \sum_{i=1}^r a_i = 1$. ■

Note. In the case of Theorem 1(a), \mathcal{L} a signed functional, if \mathcal{L} is supported on n points, then at least one of the endpoints must be included in the support of \mathcal{L} . This follows upon noting that an n th degree polynomial of norm 1 on $[0, 1]$ has at most $n + 1$ extrema in the open interval $(0, 1)$.

To distinguish the functionals described in Theorem 1, the terminology *simultaneously realizable* will be used.

DEFINITION. A linear functional $\mathcal{L} \in C^*[0, 1]$ is *simultaneously realizable (sr)* if it achieves its $C^*[0, 1]$ norm on the subspace $\mathcal{P}_n[0, 1]$. Further, if $\|\mathcal{L}\| = 1$, \mathcal{L} will be said to be *normalized sr (nsr)*.

EXAMPLE. Consider the quadratic case $n = 2$. Theorem 1(a) then yields a complete characterization of signed nsr functionals. In fact, such a signed functional must have one of the forms

- (i) $\pm [\lambda e_0 + (1 - \lambda) e_x]$, $\frac{1}{2} \leq x \leq 1, 0 < \lambda < 1$,
- (ii) $\pm [\lambda e_1 + (1 - \lambda) e_x]$, $0 \leq x \leq \frac{1}{2}, 0 < \lambda < 1$,
- (iii) $\pm [x_1 e_0 + x_2 e_{1/2} + x_3 e_1]$, $0 < x_i, \sum x_i = 1$,

In case (i), the quadratic with value 1 at 0 and minimum value -1 at x yields $\frac{1}{2} \|\mathcal{L}'\|$ (noting that $\frac{1}{2} \leq x \leq 1$ is required to yield a quadratic of norm 1). Case (ii) is analogous. In case (iii), the quadratic with values 1 at 0 and 1, and value -1 at $\frac{1}{2}$ yields $\frac{1}{2} \|\mathcal{L}'\|$.

Also in the case $n = 2$, Theorem 1(b) states that $\mathcal{L}^* = \lambda e_x + (1 - \lambda) e_y$ for some $0 \leq x \leq y \leq 1$, $0 \leq \lambda \leq 1$.

The following two theorems provide distinct characterizations for the infimum of the norms of gi projections. Each of these characterizations leads to a different numerical procedure for determining this infimum, as exemplified in Section 3, where the quadratic case $n = 2$ is discussed.

THEOREM 2. $\inf \|\mathcal{P}^{\text{gi}p}\| = \inf \|\sum_{i=0}^n v_i\|_x$, where the second infimum ranges over all $v_0, \dots, v_n \in \mathcal{P}_n[0, 1]$ adjusted so that the dual basis functionals $\mathcal{L}'_0, \dots, \mathcal{L}'_n \in \mathcal{P}_n^*[0, 1]$ (i.e., $\mathcal{L}'_i v_j = \delta_{ij}$) have norm 1.

Proof. Given such $v_0, \dots, v_n \in \mathcal{P}_n[0, 1]$, it will be shown that $\|\sum_{i=0}^n v_i\|_x$ is the limit of norms of gi projections. According to Theorem 1, extend each \mathcal{L}'_i to $\mathcal{L}'_i = \sum_{j=0}^n a_{ij} e_{x_{ij}}$, where $\|\mathcal{L}'_i\| = \sum_{j=0}^n |a_{ij}| = 1$. If there is overlap of support, let $x'_{ij} = x_{ij} + \epsilon_{ij}$ so that the supports of $\mathcal{L}''_i = \sum_{j=0}^n a_{ij} e_{x'_{ij}}$ are disjoint and in $[0, 1]$. Consider $v'_0, \dots, v'_n \in \mathcal{P}_n[0, 1]$, where $\mathcal{L}''_i v'_j = \delta_{ij}$. Then $P' = \sum_{i=0}^n \mathcal{L}''_i \otimes v'_i$ is a generalized interpolating projection. By Proposition 1.1, $\|P'\| = \|\sum_{i=0}^n v'_i\|_x$, which approaches $\|\sum_{i=0}^n v_i\|_x$ as $\epsilon_{ij} \rightarrow 0$ ($0 \leq i, j \leq n$).

On the other hand, if $P = \sum_{i=0}^n \mathcal{L}_i \otimes v_i$, where the \mathcal{L}_i have disjoint supports ($\|\mathcal{L}'_i\| = 1$), then $\|P\| = \|\sum_{i=0}^n v_i\|_x$ by Proposition 1. ■

Remark. Note that, in the first part of the proof of Theorem 2, if $P = \sum_{i=0}^n \mathcal{L}_i \otimes v_i$, then it is not in general true that $\|P'\| \rightarrow \|P\|$. Therefore, we cannot conclude that the infimum is attained. In the quadratic case, however, the infimum is attained (see Section 3).

THEOREM 3. $\inf \|\mathcal{P}^{\text{gi}p}\| = \inf \max_{p \in \mathcal{P}_n[0, 1], \|\mathcal{L}'_i p\| = 1} \|p\|_x$, where the second infimum ranges over all linearly independent nsr functionals $\mathcal{L}'_0, \dots, \mathcal{L}'_n$.

Proof. It will first be shown that

$$\inf \|\mathcal{P}^{\text{gi}p}\| = \sup_{\epsilon} \inf \max_{\substack{p \in \mathcal{P}_n[0, 1] \\ \|\mathcal{L}'_i p\| \leq 1}} \|p\|_x, \quad (\dagger)$$

where the second infimum is taken over all $\mathcal{L}'_0, \dots, \mathcal{L}'_n$ which are disjoint functionals of norm 1 in $C^*[0, 1]$, and such that, if $\mathcal{L}'_i = \mathcal{L}'_i|_{\mathcal{P}_n[0, 1]}$, then $\|\mathcal{L}'_i\| \geq 1 - \epsilon$; call these functionals nsr(ϵ) functionals. The theorem is obtained by letting $\epsilon \rightarrow 0$, and observing that subsequential limits of nsr(ϵ) functionals are nsr functionals, and that on $\mathcal{P}_n[0, 1]$ the possible loss of disjointness in the limit has no effect on $\max \|p\|_x$.

To show (†), note that any gi projection written as $P = \sum_{i=0}^n \mathcal{L}_i \otimes v_i$ (with $\|\mathcal{L}_i\| = 1$) has norm

$$\|P\| = \max_{\substack{p \in \mathcal{P}_n[0,1] \\ \mathcal{L}_i p_i = 1}} \|p\| = \left\| \sum_{i=0}^n v_i \right\|_1,$$

the equalities following from Propositions 1.1 and 1.2, respectively. By the proof of Theorem 2 it follows that $\|\sum_{i=0}^n \|\mathcal{L}_i\| \cdot \|v_i\|_1$ is a limit of norms of gi projections with nsr (ϵ) functionals. But

$$\left\| \sum_{i=0}^n \|\mathcal{L}_i\| \cdot \|v_i\|_1 \right\|_1 \leq \left\| \sum_{i=0}^n v_i \right\|_1 = \|P\|. \quad \blacksquare$$

3. MINIMAL NORM IN THE QUADRATIC CASE $n = 2$

In this section, attention will be confined to the case $n = 2$ and, in particular, to the computation of the infimum of the norms of all symmetric gi projections by use of the two different procedures inherent in Theorems 2 and 3. As the computation will show, this infimum is actually attained by a gi projection.

To facilitate the computation, and to reduce the number of parameters involved, the following lemmas are needed.

LEMMA 1. Let $\mathcal{L} \in \mathcal{P}_2^*[0, 1]$ be represented by $\mathcal{L} = a_1 e_0 + a_2 e_{1/2} + a_3 e_1$. Then

- (i) $\|\mathcal{L}\| = \sum_{i=1}^3 \|a_i\|$ if $a_1 a_3 \geq 0$ or $a_2 = 0$;
- (ii) $\|\mathcal{L}\| = a_1 + a_2 + a_3 - 2 \min\{0, a_3 + a_1 a_2 (4a_1 + a_2)^{-1}\}$ if $0 < a_1, 0 < a_2, a_3 < 0$.

(The remaining cases may be determined from (ii) by symmetry.)

Proof. In case (i), $\|\mathcal{L}\| = \sup \mathcal{L} p_2$ over all p_2 in the unit ball of $\mathcal{P}_2[0, 1]$, and is yielded by either $p_2 = \pm 1$, $p_2 = \pm (1 + 2x)$, or $p_2 = \pm [1 + 8x(x + 1)]$.

In case (ii), simple considerations show that $\mathcal{L} p_2$ is largest if p_2 is concave downward and achieves its norm (1) as a maximum value. Let

$$p_{\rho, \theta}(x) = 1 - \rho(x - \theta)^2, \quad 0 \leq \rho \leq 2/(1 - \theta)^2, \quad 0 \leq \theta \leq \frac{1}{2},$$

represent an arbitrary quadratic having these characteristics. Then $\mathcal{L} p_{\rho, \theta} = \sup_{\rho, \theta} \mathcal{L} p_{\rho, \theta}$, where

$$\mathcal{L} p_{\rho, \theta} = a_1 + a_2 + a_3 - \rho[a_1 \theta^2 + a_2(\frac{1}{2} - \theta)^2 + a_3(1 - \theta)^2].$$

Hence, for each θ , $\mathcal{L}p_{\rho,\theta}$ is linear in ρ , and the extremum is achieved for either $\rho = 0$ or $\rho = 2/(1 - \theta)^2$. Thus one has

$$\|\mathcal{L}\| = a_1 + a_2 + a_3 - 2 \min\{0, \inf_{\theta} [a_1\theta^2 + a_2(\frac{1}{2} - \theta)^2 + a_3(1 - \theta)^2] \times (1 - \theta)^{-2}\}.$$

The result follows upon differentiation of the expression in θ , noting that the minimum occurs at $\theta = a_2/(4a_1 + 2a_2)$, and simplifying the resulting expression. ■

A symmetric projection can be written as $P = \sum_{i=1}^3 \mathcal{L}_i \otimes v_i$, where $\mathcal{L}_1 f(\cdot) = \mathcal{L}_3 f(1 - \cdot)$ and $\mathcal{L}_2 f(\cdot) = \mathcal{L}_2 f(1 - \cdot)$. Hence, if $\mathcal{L}_i|_{\mathcal{P}_2} = \mathcal{L}_i = a_{i1}e_0 + a_{i2}e_{1/2} + a_{i3}e_1$, then one has $a_{1j} = a_{3(4-j)}$ for $j = 1, 2, 3$, and $a_{21} = a_{23}$. Thus, for a symmetric gip projection, the triple $\mathcal{L}_1, \mathcal{L}_2, \mathcal{L}_3$ (all of norm 1) can be represented by a matrix

$$\begin{pmatrix} a & b & c \\ d & 1 - 2|d| & d \\ c & b & a \end{pmatrix}, \quad (*)$$

where $|d| \leq \frac{1}{2}$, and $\|\mathcal{L}_1\| = 1$ determines c as a function of a and b (according to Lemma 1). Let $v_1, v_2, v_3 \in \mathcal{P}_2$ be dual to the functionals $\mathcal{L}_1, \mathcal{L}_2, \mathcal{L}_3$ ($\mathcal{L}_i v_j = \delta_{ij}$). From the symmetry of the \mathcal{L}_i , one has $v_1(x) = v_3(1 - x)$ and $v_2(x) = v_2(1 - x)$ for $0 \leq x \leq 1$.

Lemma A will provide the gross estimates which facilitate the proofs of Lemmas 2, 3, and 4. Recalling that the interpolating projection P_I carried on $\{0, \frac{1}{2}, 1\}$ has norm $5/4$, we now prove the following "continuity" result. Simple considerations of symmetry show that one can assume $a \geq |c|$.

LEMMA A. *If P is a symmetric gip onto \mathcal{P}_2 with norm $\leq 5/4$, then P is "close to" P_I in the sense that, in (*), $a \geq 4/5$, $|c| \leq 1/5$, $|b| \leq 2/5$, $|d| \leq 1/10$. Also, $a + \frac{1}{2}|b| + |c| \leq 1$.*

Proof. If $a - c < 4/5$, then consider $p \in \mathcal{P}_2$ satisfying $p(\frac{1}{2}) = 0$, $p(1) = 1/(a - c)$ and $p(0) = 1/(c - a)$. Then $\mathcal{L}_1 p = 1$, $\mathcal{L}_2 p = 0$, $\mathcal{L}_3 p = -1$; and hence $\|P\| \geq \|p\|_{\infty} = 1/(a - c) > 5/4$. This contradiction gives $a - c \geq 4/5$. A simple calculation using Lemma 1 then shows that $|b| \leq 2/5$.

We now show that $|d| \leq 1/10$, i.e., $1 - 2|d| \geq 4/5$. If $1 - 2|d| < 4/5$, then consider $p \in \mathcal{P}_2$ satisfying $p(0) = p(1) = 0$ and $p(\frac{1}{2}) = \min(5/2, 1/(1 - 2|d|))$. But $\mathcal{L}_1 p = \mathcal{L}_3 p = bp(\frac{1}{2})$ while $\mathcal{L}_2 p = (1/(1 - 2|d|))p(\frac{1}{2})$. Thus $\|P\| \geq \|p\|_{\infty} > 5/4$, since $|\mathcal{L}_i p| \leq 1$, $i = 1, 2, 3$.

Consider now $p \in \mathcal{P}_2$ satisfying $p(0) = 1/a$, $p(\frac{1}{2}) = p(1) = 0$. Then $\mathcal{L}_1 p = 1$, $\mathcal{L}_3 p = c/a$, $\mathcal{L}_2 p = d/a$. Now if $a < 1/10$, then $|c| < 1/10$ and since $|b| < 2/5$, this contradicts $\|\mathcal{L}\| = 1$. Thus $a \geq 1/10$; and since $|d| \leq 1/10$, we have $\|\mathcal{L}_2 p\| \leq 1$. Recall that $|c| \leq a$; hence also $\|\mathcal{L}_3 p\| \leq 1$. We conclude that

$a \geq 4/5$, since otherwise $\|P\| \geq \|p\|_\infty \geq 5/4$. But also since $\|\hat{\mathcal{L}}_1\| = 1$, we have that $a + \|c\| \leq 1$, and hence $\|c\| \leq \frac{1}{5}$.

Finally the conclusion that $a + \frac{1}{2}\|b\| + \|c\| \leq 1$ follows from $\|\hat{\mathcal{L}}_1 p\| \leq 1$ for all $p \in \mathcal{P}_2$ of norm 1. ■

LEMMA 2. If $\inf\|p_{\text{symm}}^{\text{tip}}\| = \frac{1}{2}\|\sum_{i=1}^3 v_i\|_x$, then $\mathcal{L}_2 = e_{1,2}$ ($d = 0$) and $0 \leq b$ in (*).

Proof. Suppose $\frac{1}{2}\|\sum_{i=1}^3 v_i\|_x = \frac{1}{2}\|v_1(x_1) + v_2(x_1) + v_3(x_1)\|$ for some $x_1 \in [0, 1]$ and a particular choice of the \pm signs; and consider $p = \pm v_1 \pm v_2 \pm v_3 \in \mathcal{P}_2$. Since $v_1 - v_2 - v_3$ is symmetrical to $v_1 + v_2 - v_3$, and since we will show at the end of the proof that $v_1 + v_2 + v_3$ is dominated by the $v_1 - v_2 - v_3$ case, there are only two relevant choices for p ; namely, the symmetrical (about $\frac{1}{2}$) case $p^{(1)} = v_1 - v_2 + v_3$ corresponding to p taking values $(1, -1, 1)$ at $(\hat{\mathcal{L}}_1, \hat{\mathcal{L}}_2, \hat{\mathcal{L}}_3)$, and the nonsymmetrical case $p^{(2)} = v_1 + v_2 - v_3$ corresponding to p taking values $(1, 1, -1)$ at $(\hat{\mathcal{L}}_1, \hat{\mathcal{L}}_2, \hat{\mathcal{L}}_3)$.

Lemma A shows that $p^{(1)}$ and $p^{(2)}$ have the pictorial representations indicated below in Fig. 1. We will use this picture only for illustrative purposes.

Since $v_3(x) = v_1(1 - x)$, one has $p^{(2)}(x) = v_1(x) + v_1(1 - x) + v_2(x)$, where

$$v_1(x) = v_1(1 - x) = \alpha(x - \frac{1}{2}) \quad \text{and} \quad v_2(x) = -\beta(x - \frac{1}{2})^2 + \gamma$$

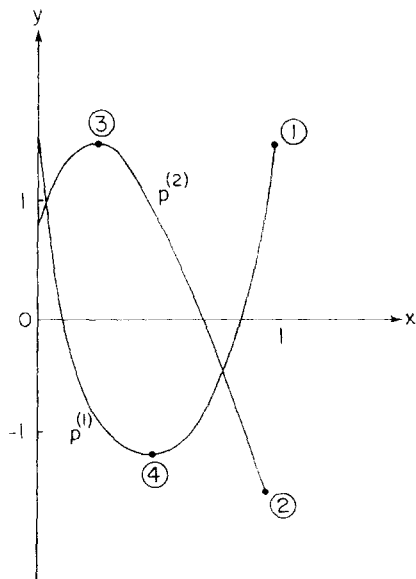


FIGURE 1.

for some α, β, γ . The determination of α, β, γ may be accomplished by using the relations $\mathcal{L}_1 p^{(2)} = 1$, $\mathcal{L}_2 p^{(2)} = 1$, and $\mathcal{L}_1 v_2 = 0$; the result being $\alpha = -2/(a+c)$, $\beta = 2r\gamma$, $\gamma = 1/(1-dr)$ (if $d \geq 0$), $\gamma = 1/(1+(4-r)d)$ (if $d < 0$), where $r = 2(a+b+c)(a-c)^{-1}$. Thus,

$$p^{(2)}(x) = \alpha(x - \tfrac{1}{2}) + \gamma[1 - 2r(x - \tfrac{1}{2})^2],$$

where γ is the *only* quantity depending on the parameter d .

Lemma A yields the gross estimates $|b| < a+c$, $|r-2| < 1$, and $|d| < 1/10$. The maximum of $p^{(2)}$, point ③ in Fig. 1, is then minimized by taking $d = 0$, i.e., $\mathcal{L}_2 = e_{1/2}$. Also, point ② has depth

$$p^{(2)}(1) = (\alpha/2) + \gamma((r/2) - 1),$$

where $\alpha < 0$ and $2 \leq r$. Hence, this depth is minimized by taking γ as small as possible, i.e., $d = 0$.

We show now that we can assume $b \geq 0$. Note that $p^{(2)}(\frac{1}{4}) = -\alpha/4 + \gamma(1-r/8)$. Since $\gamma \geq 1$, if $b < 0$, then $r < 2$ and a simple calculation shows that $a-c = 1 + b(2c+b)/(4c+b)$ yielding $a-c < 1$, and so $-\alpha > 2$. We conclude that $p^{(2)}(\frac{1}{4}) > 5/4$.

In the case of $p^{(1)}$, one has $p^{(1)}(0) = p^{(1)}(1)$; so that

$$\mathcal{L}_1 p^{(1)} = 1 \text{ yields } (a+c)p^{(1)}(1) - bp^{(1)}(\tfrac{1}{2}) = 1, \quad (*_1)$$

$$\mathcal{L}_2 p^{(1)} = -1 \text{ yields } \begin{cases} 2dp^{(1)}(1) + (1-2|d|)p^{(1)}(\tfrac{1}{2}) = -1 & (\text{if } d < 0), \\ \tfrac{1}{2}[p^{(1)}(x_0) + p^{(1)}(1-x_0)] = -1 & (\text{if } d \geq 0), \end{cases} \quad (*_2)$$

(see Theorem 2.1b). Suppose $d \geq 0$; the second condition then shows that, in order to minimize the depth of point ④, one should take $x_0 = \frac{1}{2}$ (i.e., $d = 0$). The first condition shows that, as $p^{(1)}(\frac{1}{2})$ is raised (letting $x_0 \rightarrow \frac{1}{2}$), the height of point ① is not increased. Next, suppose $d < 0$. Letting $\rho = (a+c)/(1-2|d|) - 2db$, we have $p^{(1)}(1) = (1+b-2|d|)\rho^{-1}$ and $p^{(1)}(\frac{1}{2}) = -(a+c+2d)\rho^{-1}$. A simple calculation shows that $p^{(1)}(1)$ is minimal if $d = 0$. A further simple calculation (involving several cases) shows that $|p^{(1)}(\frac{1}{2})| \leq p^{(1)}(1)$ for all d .

We now show that $v_1 + v_2 + v_3$ is dominated by $v_1 - v_2 + v_3$. In this case we obtain relations identical to $(*_1)$ and $(*_2)$ except that in $(*_2)$ $+1$'s replace -1 's on the right-hand sides of the equalities. Using $\mathcal{L}_2 = e_{1/2}$, we have that $p^{(3)}(1) = (1-b)/(a+c)$ and $p^{(3)}(\frac{1}{2}) = 1$. But $p^{(3)}(1) = (1-b)/(a+c) < (1+b)/(a+c) = p^{(1)}(1)$. ■

Note. An analysis of the whole situation in the gip quadratic case reveals that the essential factor for determining the minimal projection is a Chebyshev-type balancing between points ③ (interior maximum) and ①

(endpoint maximum) in Fig. 1; and, in fact, points (2) and (4) in the same figure are not critical.

DEFINITION. If P can be written $P = \sum_{i=0}^n \mathcal{L}_i \otimes v_i$, where all \mathcal{L}_i are positive functionals and disjoint, then P will be called a *positively representable gi projection*.

LEMMA 3. *The interpolating projection at 0, $\frac{1}{2}$, 1 is the minimal projection among all positively representable symmetric gi projections onto the quadratics.*

Proof. The proof of Lemma 2 shows that the statement of Lemma 2 also holds if the infimum is taken over merely the positively representable symmetric gi projections. Thus, one need consider only the situation.

$$\mathcal{L}_1 = \lambda e_0 + (1 - \lambda) e_z, \quad \mathcal{L}_2 = e_{1/2}, \quad \mathcal{L}_3 = \lambda e_1 + (1 - \lambda) e_{1-z},$$

where $0 < \lambda < 1$ and $z \neq 0, \frac{1}{2}$.

Consider then the case where p takes the values 1, 1, -1 at $\mathcal{L}_1, \mathcal{L}_2, \mathcal{L}_3$. As in Lemma 2, the conditions $\mathcal{L}_1 p = 1, \mathcal{L}_2 p = 1, \mathcal{L}_3 p = -1$ may be used to determine explicitly p in terms of λ and z . The interior maximum may then be computed to be

$$\max = 1 - \frac{1}{4}(r/s^2), \quad \text{where} \quad \begin{aligned} r &= (\lambda/4) + (1 - \lambda)(z - \frac{1}{2})^2, \\ s &= (1 - \lambda)(z - \frac{1}{2}) - (\lambda/2). \end{aligned}$$

Differentiating $1 - 4r/s^2$ with respect to z yields

$$d \max / dz = -\frac{1}{4} (\lambda(1 - \lambda) z / s^3),$$

which is positive since $s = -[\lambda(\frac{1}{2}) + (1 - \lambda)(\frac{1}{2} - z)] < 0$. Hence, even the interior extremum decreases to 5/4 as $z \rightarrow 0$. ■

LEMMA 4. *The interpolating projection at 0, $\frac{1}{2}$, 1 is the minimal projection among all symmetric gi projections where \mathcal{L}_1 is signed and $\hat{\mathcal{L}}_1$ has the form $\lambda e_0 - \mu e_{1/2} + (1 - \lambda - \mu) e_1$, with $0 < \lambda, \mu, 1 - \lambda - \mu$ and $2\lambda + \mu \neq 1$ (independence of $\hat{\mathcal{L}}_1, \hat{\mathcal{L}}_2, \hat{\mathcal{L}}_3$).*

Proof. The proof of Lemma 2 shows that the statement of Lemma 2 also holds if the infimum is taken over all symmetric gi projections having \mathcal{L}_1 as described. Thus, one need consider only the situation

$$\begin{aligned} \hat{\mathcal{L}}_1 &= \lambda e_0 - \mu e_{1/2} + (1 - \lambda - \mu) e_1, \\ \hat{\mathcal{L}}_2 &= e_{1/2}, \\ \hat{\mathcal{L}}_3 &= (1 - \lambda - \mu) e_0 + \mu e_{1/2} + \lambda e_1. \end{aligned}$$

As in Lemma 3, it will suffice to consider the case where p takes the values 1, $\frac{1}{2}$, -1 at $\mathcal{L}_1, \mathcal{L}_2, \mathcal{L}_3$. Using these conditions, p may be computed as

$$p(x) = 1 + \alpha(x - \frac{1}{2}) + \beta(x - \frac{1}{2})^2,$$

$$\alpha = \frac{2}{1 - 2\lambda - \mu}, \quad \beta = \frac{4(1 - 2\mu)}{1 - \mu}.$$

If $\mu = \frac{1}{2}$, p is linear with $\|p\|_\infty$ achieved at either 0 or 1. But, in this case,

$$p(0) = 1 - (\alpha/2), \quad p(1) = 1 + (\alpha/2);$$

which gives

$$\|p\|_\infty = 1 + (|\alpha|/2) = 1 + (1/|1 - 2\lambda - \mu|) \geq 2.$$

If $\mu \neq \frac{1}{2}$, the extreme values of p are given by

$$1 + \frac{\alpha}{2} + \frac{\beta}{4} = \frac{\mu}{1 - \mu} \pm \frac{1}{1 - 2\lambda - \mu} \quad \text{at } x = 0, 1;$$

$$1 + \frac{\alpha^2}{4\beta} = 1 + \frac{1 - \mu}{4(1 - 2\mu)(1 - 2\lambda - \mu)^2} \quad \text{at } x_{\text{int}} = \frac{1}{2} + \frac{\alpha}{2\beta}$$

(if $x_{\text{int}} \in [0, 1]$).

A close analysis of these extrema shows that $\|p\|_\infty \geq 5/4$. ■

Numerical procedure 1. Theorems 2.1 and 2.2 provide that the infimum of the norms of all symmetric gi projections onto the quadratics may be obtained as follows. Invert the matrix

$$\begin{pmatrix} a & b & c \\ 0 & 1 & 0 \\ c & b & a \end{pmatrix}, \quad \begin{aligned} 4/5 &\leq a \leq 1 \\ 0 &\leq b \leq \frac{1}{2}[1 - 3a + (1 + 10a - 7a^2)^{1/2}] \\ c &= a + b - 2ab(4a + b)^{-1} - 1 \end{aligned} \quad (**)$$

to obtain

$$(v_{ij}) = (1/(a^2 - c^2)) \begin{pmatrix} a & -b(a - c) & -c \\ 0 & a^2 - c^2 & 0 \\ -c & -b(a - c) & a \end{pmatrix}.$$

(Lemmas 1, 2, 3, 4 guarantee that the restrictions in (**) are admissible.) The columns of this last matrix provide the values of v_j at 0, $\frac{1}{2}$, 1, i.e.,

$$v_j((i - 1)/2) = v_{ij}, \quad i = 1, 2, 3 \text{ and } j = 1, 2, 3.$$

If

$$N(a, b) = \sum_{j=1}^3 v_j = \max_{\{v_j\}} \{v_1 + v_2 + v_3\},$$

then

$$\inf_{\{P\}} \|P_{\text{symm}, 1}^{\text{gip}}\| = \inf_{a, b} N(a, b).$$

This last infimum was determined by means of a two-parameter (a and b) search technique on a Hewlett-Packard 9830A programmable calculator. For the results, see Theorem 1 below.

Numerical procedure II. Theorems 2.1 and 2.3 provide that the infimum of the norms of all symmetric gip projections onto the quadratics may be obtained as follows. For $0 \leq \lambda < 1$ and $0 \leq z < \frac{1}{2}$, consider the functionals

$$\mathcal{L}_1 = \lambda e_1 + (1 - \lambda) e_1, \quad \mathcal{L}_2 = e_{1+2}, \quad \mathcal{L}_3 = \lambda e_{1+2} + (1 - \lambda) e_n.$$

(Lemmas 1, 2, 3, 4 guarantee that the indicated restrictions on $\mathcal{L}_1, \mathcal{L}_2, \mathcal{L}_3$ are admissible.) If

$$N(\lambda, z) = \max_{\{P\}} \|P\| = \max_{\{p\}} \|p\|_{\mathcal{L}_i} p = \|1\|_{\mathcal{L}_i}.$$

then

$$\inf_{\{P\}} \|P_{\text{symm}, 1}^{\text{gip}}\| = \inf_{\lambda, z} N(\lambda, z).$$

Again, a two-parameter search technique was used to obtain the results in the following theorem.

THEOREM 1. *In the quadratic case,*

$$\inf_{\{P\}} \|P_{\text{symm}, 1}^{\text{gip}}\| = 1.24839.$$

The infimum is uniquely (see the following note) achieved for $P = \sum_{i=1}^3 \mathcal{L}_i \otimes v_i$, where

$$\mathcal{L}_1 = 0.94876 e_{0.014322} + 0.05124 e_1,$$

$$\mathcal{L}_2 = e_{1+2},$$

$$\mathcal{L}_3 = 0.94876 e_{0.985678} + 0.05124 e_0.$$

Note. Uniqueness is obtained in the following sense. For each fixed z , a convex function of λ is being minimized. The resulting function of the single variable z shows, to the accuracy of the computation, a unique minimum.

Remark. While the above numerical procedures were of equal difficulty, the situation would appear to be different for $n > 2$. Procedure I required the

calculation of the norm (on \mathcal{P}_3) of each functional \mathcal{L}_i . Even in the cubic case this calculation appears difficult, and would seem to indicate that Procedure II is preferable.

Remark. The restriction of considering only symmetric g_i projections has some basis in numerical experiments. Numerous random searches were conducted without using the assumption of symmetry; the result being, in each case, an indication that symmetry yielded lower norms. At this time, the authors have been unable to establish, in a rigorous way, that symmetry must hold amongst a subset of the minimal g_i projections (as is the case for the set of minimal projections: see [1]).

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